

# Gell-Mann–Okubo Mass Formula for $SU(4)$ Meson Hexadecuplet

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## Abstract

Using a linear mass spectrum of an  $SU(4)$  meson hexadecuplet, we derive the Gell-Mann–Okubo mass formula for the charmed mesons, in good agreement with experiment. Possible generalization of this method to a higher symmetry group is briefly discussed.

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The hadronic mass spectrum is an essential ingredient in theoretical investigations of the physics of strong interactions. It is well known that the correct thermodynamic description of hot hadronic matter requires consideration of higher mass excited states, the resonances, whose contribution becomes essential at temperatures  $\sim O(100 \text{ MeV})$  [1, 2]. The method for taking into account these resonances was suggested by Belenky and Landau [3] as considering unstable particles on an equal footing with the stable ones in the thermodynamic quantities; e.g., the formulas for the pressure and energy density in a resonance gas read<sup>1</sup>

$$p = \sum_i p_i = \sum_i g_i \frac{m_i^2 T^2}{2\pi^2} K_2 \left( \frac{m_i}{T} \right), \quad (1)$$

$$\rho = \sum_i \rho_i, \quad \rho_i = T \frac{dp_i}{dT} - p_i, \quad (2)$$

where  $g_i$  are the corresponding degeneracies ( $J$  and  $I$  are spin and isospin, respectively),

$$g_i = \frac{\pi^4}{90} \times \begin{cases} (2J_i + 1)(2I_i + 1) & \text{for non-strange mesons} \\ 4(2J_i + 1) & \text{for strange (K) mesons} \\ 2(2J_i + 1)(2I_i + 1) \times 7/8 & \text{for baryons} \end{cases}$$

These expressions may be rewritten with the help of a *resonance spectrum*,

$$p = \int_{m_1}^{m_2} dm \tau(m) p(m), \quad p(m) \equiv \frac{m^2 T^2}{2\pi^2} K_2 \left( \frac{m}{T} \right), \quad (3)$$

$$\rho = \int_{m_1}^{m_2} dm \tau(m) \rho(m), \quad \rho(m) \equiv T \frac{dp(m)}{dT} - p(m), \quad (4)$$

normalized as

$$\int_{m_1}^{m_2} dm \tau(m) = \sum_i g_i, \quad (5)$$

where  $m_1$  and  $m_2$  are the masses of the lightest and heaviest species, respectively, entering the formulas (1),(2).

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<sup>1</sup>For simplicity, we neglect the chemical potential and approximate the particle statistics by the Maxwell-Boltzmann one.

In both the statistical bootstrap model [4, 5] and the dual resonance model [6], a resonance spectrum takes on the form

$$\tau(m) \sim m^a e^{m/T_0}, \quad (6)$$

where  $a$  and  $T_0$  are constants. The treatment of a hadronic resonance gas by means of the spectrum (6) leads to a singularity in the thermodynamic functions at  $T = T_0$  [4, 5] and, in particular, to an infinite number of the effective degrees of freedom in the hadron phase, thus hindering a transition to the quark-gluon phase. Moreover, as shown by Fowler and Weiner [7], an exponential mass spectrum of the form (6) is incompatible with the existence of the quark-gluon phase: in order that a phase transition from the hadron phase to the quark-gluon phase be possible, the hadronic spectrum cannot grow with  $m$  faster than a power.

In our previous work [8] we considered a model for a transition from a phase of strongly interacting hadron constituents, described by a manifestly covariant relativistic statistical mechanics which turned out to be a reliable framework in the description of realistic physical systems [9], to the hadron phase described by a resonance spectrum, Eqs. (3),(4). An example of such a transition is what might be considered to be a relativistic high temperature Bose-Einstein condensation studied by the authors in ref. [10], which corresponds, in the way suggested by Haber and Weldon [11], to spontaneous flavor symmetry breakdown,  $SU(3)_F \rightarrow SU(2)_I \times U(1)_Y$ , upon which hadronic multiplets are formed, with the masses obeying the Gell-Mann–Okubo formulas [12]

$$m^\ell = a + bY + c \left[ \frac{Y^2}{4} - I(I+1) \right]; \quad (7)$$

here  $I$  and  $Y$  are the isospin and hypercharge, respectively,  $\ell$  is 2 for mesons and 1 for baryons, and  $a, b, c$  are independent of  $I$  and  $Y$  but, in general, depend on  $(p, q)$ , where  $(p, q)$  is any irreducible representation of  $SU(3)$ . Then only the assumption on the overall degeneracy being conserved during the transition is required to lead to the unique form of a resonance spectrum in the hadron phase:

$$\tau(m) = Cm, \quad C = \text{const.} \quad (8)$$

Zhirov and Shuryak [13] have found the same result on phenomenological grounds. As shown in ref. [13], the spectrum (8), used in the formulas (3),(4) (with the upper limit of integration infinity), leads to the equation of state  $p, \rho \sim T^6$ ,  $p = \rho/5$ , called by Shuryak the “realistic” equation of state for hot hadronic matter [1], which has some experimental support. Zhirov and Shuryak [13] have calculated the velocity of sound,  $c_s^2 \equiv dp/d\rho = c_s^2(T)$ , with  $p$  and  $\rho$  defined in Eqs. (1),(2), and found that  $c_s^2(T)$  at first increases with  $T$  very quickly and then saturates at the value of  $c_s^2 \simeq 1/3$  if only the pions are taken into account, and at  $c_s^2 \simeq 1/5$  if resonances up to  $M \sim 1.7$  GeV are included.

We have checked the coincidence of the results given by the linear spectrum (8) with those obtained directly from Eq. (1) for the actual hadronic species with the corresponding degeneracies, for all well-established multiplets, both mesonic and baryonic,  $f'_2(1525)$ , and found it excellent [8]. Therefore, the theoretical conclusion that a linear spectrum is the actual spectrum in the description of individual hadronic multiplets finds its experimental confirmation as well. In our recent paper [14] we applied a linear spectrum to the problem of establishing the correct  $q\bar{q}$  assignment for the problematic meson nonets, like the scalar, axial-vector and tensor ones, and separating out non- $q\bar{q}$  mesons.

The easiest way to see that a linear spectrum corresponds to the actual spectrum of a meson nonet is as follows<sup>2</sup>. Let us calculate the average mass squared for a spin- $s$  nonet:

$$\langle m^2 \rangle_9 \equiv \frac{\sum_i g_i m_i^2}{\sum_i g_i} = \frac{3m_1^2 + 4m_{1/2}^2 + m_{0'}^2 + m_{0''}^2}{9}, \quad (9)$$

where  $m_1$ ,  $m_{1/2}$ ,  $m_0$ ,  $m_{0'}$  are the masses of isovector, isospinor, and two isoscalar states, respectively, and the spin degeneracy,  $2s + 1$ , cancels out. In general, the isoscalar states<sup>3</sup>  $\omega_{0'}$  and  $\omega_{0''}$  are the octet  $\omega_8$  and singlet  $\omega_0$  mixed states because of  $SU(3)$  breaking,

$$\omega_{0'} = \omega_8 \cos \theta_M - \omega_0 \sin \theta_M,$$

$$\omega_{0''} = \omega_8 \sin \theta_M + \omega_0 \cos \theta_M,$$

where  $\theta_M$  is a mixing angle. Assuming that the matrix element of the Hamiltonian between the states yields a mass squared, i.e.,  $m_{0'}^2 = \langle \omega_{0'} | H | \omega_{0'} \rangle$  etc., one obtains from the above relations [15],

$$m_{0'}^2 = m_8^2 \cos^2 \theta_M + m_0^2 \sin^2 \theta_M - 2m_{08}^2 \sin \theta_M \cos \theta_M, \quad (10)$$

$$m_{0''}^2 = m_8^2 \sin^2 \theta_M + m_0^2 \cos^2 \theta_M + 2m_{08}^2 \sin \theta_M \cos \theta_M. \quad (11)$$

Since  $\omega_{0'}$  and  $\omega_{0''}$  are orthogonal, one has further

$$m_{0'0''}^2 = 0 = (m_8^2 - m_0^2) \sin \theta_M \cos \theta_M + m_{08}^2 (\cos^2 \theta_M - \sin^2 \theta_M). \quad (12)$$

Eliminating  $m_0$  and  $m_{08}$  from (10)-(12) yields

$$\tan^2 \theta_M = \frac{m_8^2 - m_{0'}^2}{m_{0''}^2 - m_8^2}. \quad (13)$$

It also follows from (10),(11) that, independent of  $\theta_M$ ,  $m_{0'}^2 + m_{0''}^2 = m_8^2 + m_0^2$ , and therefore, Eq. (9) may be rewritten as

$$\langle m^2 \rangle_9 = \frac{3m_1^2 + 4m_{1/2}^2 + m_8^2 + m_0^2}{9}. \quad (14)$$

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<sup>2</sup>For a baryon multiplet, it is more difficult to show that the mass spectrum is linear, since the Gell-Mann–Okubo formulas are linear in mass for baryons. More detailed discussion is given in [8].

<sup>3</sup>The  $\omega_{0'}$  is a mostly octet isoscalar.

For the octet,  $(3\ m_1, 4\ m_{1/2}, 1\ m_8)$ , the Gell-Mann–Okubo formula (as follows from (7)) is

$$4m_{1/2}^2 = 3m_8^2 + m_1^2. \quad (15)$$

Therefore, the average mass squared for the octet is

$$\langle m^2 \rangle_8 = \frac{3m_1^2 + 4m_{1/2}^2 + m_8^2}{8} = \frac{m_1^2 + m_8^2}{2}, \quad (16)$$

where Eq. (15) was used. In the exact  $SU(3)$  limit where the  $u$ ,  $d$  and  $s$  quarks have equal masses, all the squared masses of the nonet states are equal as well. Since in this limit all the squared masses of the octet states are equal to the average mass squared of the octet<sup>4</sup>, Eq. (16), the mass of the singlet should have the same value,<sup>5</sup> i.e.,

$$m_0^2 = \frac{m_1^2 + m_8^2}{2}. \quad (17)$$

With Eq. (15), it then follows from (17) that

$$m_0^2 + m_8^2 = 2m_{1/2}^2,$$

which reduces, through  $m_0^2 + m_8^2 = m_{0'}^2 + m_{0''}^2$ , to

$$m_{0'}^2 + m_{0''}^2 = 2m_{1/2}^2, \quad (18)$$

which is an extra Gell-Mann–Okubo mass relation for a nonet. We have checked this relation in a separate paper [18], and found that with the experimentally available meson masses, the relative error in the values on the l.h.s. and r.h.s. of Eq. (18) does not exceed 3% for all well-established nonets (except for the pseudoscalar nonet for

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<sup>4</sup>In a manifestly covariant theory, this holds since a total mass squared is rigorously conserved. In the standard framework, for pseudoscalar mesons, this is easily seen by using the lowest order relations [16]  $m_1^2 \equiv m_\pi^2 = 2mB$ ,  $m_{1/2}^2 \equiv m_K^2 = (m + m_s)B$ , where  $m = (m_u + m_d)/2$ , and  $B$  is related to the quark condensate. Therefore, it follows from (15),(16) that  $m_8^2 = 2/3 (2m_s + m)B$ ,  $\langle m^2 \rangle_8 = 2/3 (m_s + 2m)B = 2/3 (m_u + m_d + m_s)B$ . In the exact  $SU(3)$  limit,  $m_u = m_d = m_s = \bar{m}$ , and hence  $m_1^2 = m_{1/2}^2 = m_8^2 = \langle m^2 \rangle_8 = 2\bar{m}B$ . For higher mass mesons, since the states with equal isospin (and alternating parity) lie on linear Regge trajectories, one may expect the relations of the form  $(c = C/B)$   $m_1^2 = 2mB + C = (2m + c)B$ ,  $m_{1/2}^2 = (m + m_s)B + C = (m + m_s + c)B$ ,  $m_8^2 = 2/3 (2m_s + m)B + C = 2/3 (2m_s + m + 3/2 c)B$ , consistent with the Gell-Mann–Okubo formula (15), leading to  $m_1^2 = m_{1/2}^2 = m_8^2 = \langle m^2 \rangle_8 = 2\bar{m}B + C$  in the  $SU(3)$  limit  $m_u = m_d = m_s = \bar{m}$ . For vector mesons, such a relation was obtained by Balázs in the flux-tube fragmentation approach to a low-mass hadronic spectrum [17],  $m_\rho^2 = m_\pi^2 + 1/2\alpha'$ , with  $\alpha'$  being a universal Regge slope, in good agreement with the experiment.

<sup>5</sup>It is also seen from the relations of a previous footnote: since the total mass squared of a nonet is proportional to the total mass of quarks the nonet members are made of,  $\sum_i g_i m_i^2 = (12m + 6m_s)B + 9C$ , it follows from the above expressions for  $m_1^2$ ,  $m_{1/2}^2$  and  $m_8^2$  that  $m_0^2 = 2/3 (2m + m_s)B + C = \langle m^2 \rangle_8 = \langle m^2 \rangle_9$ .

which Eq. (18) does not hold, perhaps because the  $\eta_0$  develops a large dynamical mass due to axial  $U(1)$  symmetry breakdown before it mixes with the  $\eta_8$  to form the physical  $\eta$  and  $\eta'$  states). For a singlet-octet mixing close to “ideal” one,  $\tan \theta_M \simeq 1/\sqrt{2}$ ; it then follows from (13) that

$$2m_{0'}^2 + m_{0''}^2 \simeq 3m_8^2,$$

which reduces, through (15),(18), to

$$m_{0''} \simeq m_1. \quad (19)$$

Now it follows clearly that the ground states of all well-established nonets<sup>6</sup> (except for the pseudoscalar one) are almost mass degenerate pairs, like  $(\rho, \omega)$ .<sup>7</sup> In the close-to-ideal mixing case, Eq. (18) may be rewritten, with the help of (19), as

$$m_1^2 + m_{0'}^2 \simeq 2m_{1/2}^2. \quad (20)$$

This relation for pseudoscalar and vector mesons with the ground states being the mass degenerate pairs  $(\pi, \eta_0)$  and  $(\rho, \omega)$ , respectively, was previously obtained by Balázs and Nicolescu using the dual-topological-unitarization approach to the confinement region of hadronic physics (Eq. (21) of ref. [19]). With (16) and (17), Eq. (10) finally reduces to

$$\langle m^2 \rangle_9 = \frac{m_1^2 + m_8^2}{2}, \quad (21)$$

which, of course, coincides with both,  $\langle m^2 \rangle_8$  in (16) and  $m_0^2$  in (17), which is the property of the  $SU(3)$  limit (or the conservation of a total mass squared in a manifestly covariant theory).

For the actual mass spectrum of the nonet, the average mass squared (9) may be represented in the form<sup>8</sup>

$$\langle m^2 \rangle_9 = \frac{\int_{m_1}^{m_8} dm \tau(m) m^2}{\int_{m_1}^{m_8} dm \tau(m)}, \quad (22)$$

and one sees that the only choice for  $\tau(m)$  leading to the relation (21) is  $\tau(m) = Cm$ ,  $C = \text{const.}$  Indeed, in this case

$$\langle m^2 \rangle_9 = \frac{\int_{m_1}^{m_8} dm m^3}{\int_{m_1}^{m_8} dm m} = \frac{(m_8^4 - m_1^4)/4}{(m_8^2 - m_1^2)/2} = \frac{m_1^2 + m_8^2}{2},$$

in agreement with (21).

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<sup>6</sup>This is also true for  $q\bar{q}$  assignment of the scalar meson nonet suggested by the authors in ref. [14].

<sup>7</sup>It follows from the relations of footnote 4 that, in the close-to-ideal mixing case,  $m_{0'}^2 \simeq 2m_s B + C$  and  $m_{0''}^2 \simeq 2mB + C = m_1^2$ .

<sup>8</sup>Since  $m_s > m$ ,  $m_1 < m_{1/2} < m_8$ , as seen in the relations of footnote 4. Moreover,  $m_1 < m_0 < m_8$ , and therefore, the range of integration in Eq. (22) is  $(m_1, m_8)$ .

Evidently, one may choose an opposite way, viz., starting from a linear spectrum as the actual spectrum of a nonet, to derive the Gell-Mann–Okubo mass formula. To this end, one should first calculate the average mass squared, Eq. (21). Then one has to place 9 nonet states in the interval  $(m_1, m_8)$  in a way that preserves the average mass squared. As we already know, the isoscalar singlet mass squared should coincide with the average mass squared; for the remaining 8 states one would have the relation (16) which would in turn reduce to the Gell-Mann–Okubo formula (15). One sees that the assumption of a linear mass spectrum turns out to be a good alternative to group theory assumptions on the form of the mass splitting, for the derivation of the Gell-Mann–Okubo type relations, which may be rather difficult technical task for a higher symmetry group.

We now generalize the above derivation of the Gell-Mann–Okubo mass formula to the case of 4 flavors, by simply adding one more quark ( $c$ -quark). Then, in addition to the 9 states of a nonet, we have to add 7 more states, to finally form an  $SU(4)$  hexadecuplet: 4  $c\bar{u}, u\bar{c}, c\bar{d}, d\bar{c}$ , 2  $c\bar{s}, s\bar{c}$ , and 1  $c\bar{c}$ . Suppose that  $SU(3)$  symmetry is exact but the underlying  $SU(4)$  one is broken by the  $c$ -quark mass. Then the 9 nonet states have equal mass squared which coincide with  $m_0^2$ , and of the remaining 7 states, 6 (which are the combinations of  $c$ -quark with one of  $u, d, s$ ) have equal masses as well. Now we have to place 7 states in the interval  $(m_0, m_{c\bar{c}})$  in a way that preserves the average mass squared ( $q$  stands for one of  $u, d, s$ ):

$$\frac{6m_{c\bar{q}}^2 + m_{c\bar{c}}^2}{7} = \frac{m_0^2 + m_{c\bar{c}}^2}{2}; \quad (23)$$

therefore

$$12m_{c\bar{q}}^2 = 5m_{c\bar{c}}^2 + 7m_0^2, \quad (24)$$

which is the generalization of the Gell-Mann–Okubo mass formula to the case of broken  $SU(4)$  but exact  $SU(3)$  symmetry.

In a real world, both,  $SU(4)$  and  $SU(3)$ , are broken, so that we have to distinguish between the masses of  $c\bar{u}, c\bar{d}$ , and  $c\bar{s}$  states. Introducing the standard notations,  $D(c\bar{u}, c\bar{d})$ ,  $D_s(c\bar{s})$ ,  $\Psi(c\bar{c})$ , we have to modify Eq. (23), as follows:

$$\frac{4m_D^2 + 2m_{D_s}^2 + m_\Psi^2}{7} = \frac{m_0^2 + m_\Psi^2}{2}; \quad (25)$$

thus

$$8m_D^2 + 4m_{D_s}^2 = 5m_\Psi^2 + 7m_0^2, \quad (26)$$

which is the Gell-Mann–Okubo mass formula for an  $SU(4)$  hexadecuplet (which has to be accompanied by those for an  $SU(3)$  nonet, Eqs. (15),(16)). Let us check this formula for the experimentally established masses of the charmed mesons, for the following multiplets (which are the only multiplets for which these mesons have been discovered experimentally)<sup>9</sup>:

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<sup>9</sup>The values of  $m_0^2$  given below are calculated from Eq. (17), which reduces, through (15), to  $m_0^2 = (m_1^2 + 2m_{1/2}^2)/3$ .

1)  $1^1S_0$   $J^{PC} = 0^{-+}$  pseudoscalar mesons,  $m(D) = 1.87$  GeV,  $m(D_s) = 1.97$  GeV,  $m(\eta_c) = 2.98$  GeV, and  $m_0^2 = 0.17$  GeV<sup>2</sup>. Therefore, one obtains 43.5 GeV<sup>2</sup> on the l.h.s. of Eq. (26), vs. 45.6 GeV<sup>2</sup> on the r.h.s.

2)  $1^3S_1$   $J^{PC} = 1^{--}$  vector mesons,  $m(D^*) = 2.01$  GeV,  $m(D_s^*) = 2.11$  GeV,  $m(J/\Psi) = 3.09$  GeV, and  $m_0^2 = 0.73$  GeV<sup>2</sup>. Therefore, one has 50.1 GeV<sup>2</sup> on the l.h.s. of Eq. (26) vs. 52.8 GeV<sup>2</sup> on the r.h.s.

3)  $1^1P_1$   $J^{PC} = 1^{+-}$  pseudovector mesons,  $m(D_1) = 2.42$  GeV,  $m(D_{s1}) = 2.54$  GeV,  $m(h_c(1P)) = 3.53$  GeV, and  $m_0^2 = 1.58$  GeV<sup>2</sup>. In this case one has 72.7 GeV<sup>2</sup> on the l.h.s of (26) vs. 73.3 GeV<sup>2</sup> on the r.h.s.

4)  $1^3P_2$   $J^{PC} = 2^{++}$  tensor mesons<sup>10</sup>,  $m(D_2^*) = 2.46$  GeV,  $m(D_{s2}^*) = 2.57$  GeV,  $m(\chi_{c2}(1P)) = 3.55$  GeV, and  $m_0^2 = 1.94$  GeV<sup>2</sup>. Now one has 74.8 GeV<sup>2</sup> on the l.h.s of (26) vs. 76.5 GeV<sup>2</sup> on the r.h.s.

Thus, in either case, the relative error does not exceed 5%, in the third case the results almost coincide.

One may try to generalize the presented method of the derivation of the Gell-Mann–Okubo formula from a linear mass spectrum to a higher symmetry group. Let us briefly discuss this point. Suppose, as previously, that  $SU(N)$  flavor symmetry is exact but the underlying  $SU(N+1)$  one is broken by the mass of the  $(N+1)$ -quark. Then we have a mass degenerate  $SU(N)$  multiplet ( $N^2$  states), and  $2N+1$  more states, of which  $2N$  are mass degenerate as well. Now we have to place these  $2N+1$  states in the mass interval  $(m_{(N,N)}, m_{(N+1,N+1)})$  in a way that preserves the average mass squared:

$$\frac{2Nm_{(N,N+1)}^2 + m_{(N+1,N+1)}^2}{2N+1} = \frac{m_{(N,N)}^2 + m_{(N+1,N+1)}^2}{2}. \quad (27)$$

This is the analog of the Gell-Mann–Okubo mass formula for an  $SU(N+1)$  multiplet. It one writes down roughly ( $m_N$  and  $m_{N+1}$  are the masses of  $N$ - and  $(N+1)$ -quarks, respectively)

$$m_{(N,N)} \simeq 2m_N, \quad m_{(N,N+1)} \simeq m_N + m_{N+1}, \quad m_{(N+1,N+1)} \simeq 2m_{N+1} \quad (28)$$

and neglects  $m_N^2$  in comparison with  $m_{N+1}^2$ , one obtains

$$m_{N+1} \simeq \frac{2N}{N-1} m_N. \quad (29)$$

Now it becomes clear why the quark masses are exactly what they are. If one goes from  $SU(3)$  to  $SU(4)$ , one obtains from (29)  $m_4 \simeq 3m_3$ , i.e.,  $m_c \simeq 3m_s$ , where  $m_s$  is the constituent  $s$ -quark mass. With  $m_s \simeq 0.5 - 0.55$  GeV, one has  $m_c \simeq 1.5 - 1.6$

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<sup>10</sup>Although the  $D_{s2}^*(2573)$  meson was omitted from the recent Meson Summary Table as “needs confirmation”, it is the best candidate on the  $1^3P_2$   $2^{++}$   $c\bar{s}$ -state since its width and decay modes are consistent with  $J^P = 2^+$  [20].



GeV, exactly the value which may be expected from the naive quark model and which is indicated by the recent Particle Data Group [21]. If one further goes from  $SU(4)$  to  $SU(5)$ , one obtains  $m_5 \simeq 8/3 m_4$ , i.e.,  $m_b \simeq 8/3 m_c$ . With  $m_c \simeq 1.5 - 1.6$  GeV, this implies  $m_b \simeq 4.0 - 4.3$  GeV, again in agreement with the value  $4.1 - 4.5$  GeV given by the recent Particle Data Group [21] and the expectations from the naive quark model. We therefore expect the corresponding Gell-Mann–Okubo formula for an  $SU(5)$  25-plet to be in a fair agreement with the experiment. Unfortunately, if one goes from  $SU(5)$  to  $SU(6)$ , one gets the value of the top-quark mass  $m_t \simeq 5/2 m_b \simeq 10 - 11$  GeV, which is one order of magnitude less than that expected from the Standard Model<sup>11</sup> [21]. The explanation of this fact may lie in the mass spectrum deviating from a linear form for the masses of the order of  $\sim O(m_b)$ . (E.g., the mass spectrum may be of the form  $\tau(m) \sim m \cdot f(m)$ , where  $f(m)$  is close to unity for the masses up to  $\sim O(m_b)$  and begins to grow rapidly for higher masses.) In fact, in a manifestly covariant theory, the mass spectrum is  $\tau(m) \sim m \cdot \exp(\alpha m^2)$ , but normally one expects  $\alpha$  to be very small. We also note that the actual mass spectrum cannot be of the Hagedorn form (6) because the exponent  $a$  in Eq. (6) is always negative (and is related to the number of transverse dimensions of a string theory [23]), and therefore cannot be equal to unity. More detailed discussion on this point, as well as the Gell-Mann–Okubo formulas for  $SU(4)$  baryon multiplets, will be given in a separate publication.

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## References

- [1] E.V. Shuryak, *The QCD Vacuum, Hadrons and the Superdense Matter*, (World Scientific, Singapore, 1988)
- [2] C.R. Alcock, G.M. Fuller and G.J. Mathews, *Astrophys. J.* **320** (1987) 439  
H. Bebie, P. Gerber, J.L. Goity and H. Leutwyler, *Nucl. Phys. B* **378** (1992) 95
- [3] S.Z. Belenky and L.D. Landau, *Sov. Phys. Uspekhi* **56** (1955) 309; *Nuovo Cim. Suppl.* **3** (1956) 15

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<sup>11</sup>We note, however, that the low-energy relation between the  $b$ - and  $t$ -quark masses obtained by Jungman by performing a detailed numerical study of the Yukawa-coupling renormalization-group flow in  $SO(10)$  models is [22]  $m_b/m_t \simeq 2.5 - 2.7$ .

- [4] R. Hagedorn, Nuovo Cim. **35** (1965) 216; Nuovo Cim. Suppl. **6** (1968) 169, 311
- [5] S. Frautschi, Phys. Rev. D **3** (1971) 2821
- [6] S. Fubini and G. Veneziano, Nuovo Cim. A **64** (1969) 811
- [7] G.N. Fowler and R.M. Weiner, Phys. Lett. B **89** (1980) 394
- [8] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Hadronic Resonance Spectrum: Power vs. Exponential Law. Experimental Evidence; to be published
- [9] L.P. Horwitz and C. Piron, Helv. Phys. Acta **46** (1973) 316  
 L.P. Horwitz, W.C. Schieve and C. Piron, Ann. Phys. (N.Y.) **137** (1981) 306  
 L.P. Horwitz, S. Shashoua and W.C. Schieve, Physica A **161** (1989) 300  
 L. Burakovsky, Manifestly Covariant Relativistic Statistical Mechanics as a Framework for Description of Realistic Physical Systems, Ph.D. thesis (Tel-Aviv University, 1995), unpublished; L. Burakovsky, L.P. Horwitz and W.C. Schieve, Mass - Proper Time Uncertainty Relation in a Manifestly Covariant Relativistic Statistical Mechanics, to be published
- [10] L. Burakovsky, L.P. Horwitz and W.C. Schieve, A New Relativistic High Temperature Bose-Einstein Condensation; Phys. Rev. D, *in press*
- [11] H.A. Haber and H.E. Weldon, Phys. Rev. D **25** (1982) 502 (1994) 4725
- [12] S. Okubo, Prog. Theor. Phys. **27** (1962) 949, **28** (1962) 24  
 M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, (Benjamin, N.Y., 1964)
- [13] O.V. Zhirov and E.V. Shuryak, Sov. J. Nucl. Phys. **21** (1975) 443
- [14] L. Burakovsky and L.P. Horwitz, Hadronic Resonance Spectrum May Help in Resolution of Meson Nonet Enigmas; to be published
- [15] D.H. Perkins, *Introduction to High Energy Physics*, 3rd ed., (Addison Wesley, Reading, Ma, 1987), section 5.6
- [16] M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. **175** (1968) 2195
- [17] L.A.P. Balázs, Phys. Lett. B **120** (1983) 426
- [18] L. Burakovsky and L.P. Horwitz, Mass Spectrum of a Meson Nonet is Linear, in preparation
- [19] L.A.P. Balázs and B. Nicolescu, Phys. Rev. D **28** (1983) 2818
- [20] Particle Data Group, Phys. Rev. D **50** (1994) 1600

- [21] Particle Data Group, Phys. Rev. D **50** (1994) 1195
- [22] G. Jungman, Phys. Rev. D **46** (1992) 4004, **47** (1993) 346
- [23] K.R. Dienes and J.-R. Cudell, Phys. Rev. Lett. **72** (1994) 187